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Green's Functions in Broken Discrete Symmetry

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Abstract

Scalar fields with interaction

$$\lambda \sum_{i \neq j} \phi_i^2 \phi_j^2 \qquad \text{or} \qquad \lambda' \sum_{i, j, k, l \text{ pairwise distinct}} \phi_i \phi_j \phi_k \phi_l$$

are considered in Green's function formalism of quantum field theory. Equations for Green's functions are derived under the assumption that some G_{2n+1} are not zero. Descending problems from given (model) G_3, G_4 and from G_4, G_5 are discussed, without appealing to perturbation theory.

1. *Introduction*

In this paper we consider symmetry-breaking solutions of Green's function equations derived from Lagrangians without continuous internal symmetry. As Goldstone's theorem applies only to breakings of continuous groups, one need not worry about unwanted massless bosons in the above-stated problem.

To be specific let us begin with the following Lagrangian:

$$\mathscr{L} = \frac{1}{2} \sum_{i=1}^{N} \left[\left(\frac{\partial \phi_i}{\partial x_\mu} \right)^2 - \mu_i^2 \phi_i^2 \right] + \frac{1}{2} g \sum_{i \neq j} \phi_i^2 \phi_j^2, \qquad N = 4,5,6 \quad (1.1)$$

The only symmetry groups apart from the Lorentz group are discrete groups of involutions and permutations. If it is assumed that the Green's function inherit the invariance under those groups of the Lagrangian, the equations to be considered are those shown graphically in Figure 1. If one wants to look for a set of Green's functions that does not inherit the invariance of the Lagrangian, one has to consider the following equations (in momentum representation):

$$G_{2,i}(p) = \{p^2 - \mu_i^2 - \Sigma_i^{\text{reg}}(p)\}^{-1} \qquad (1.2a)$$

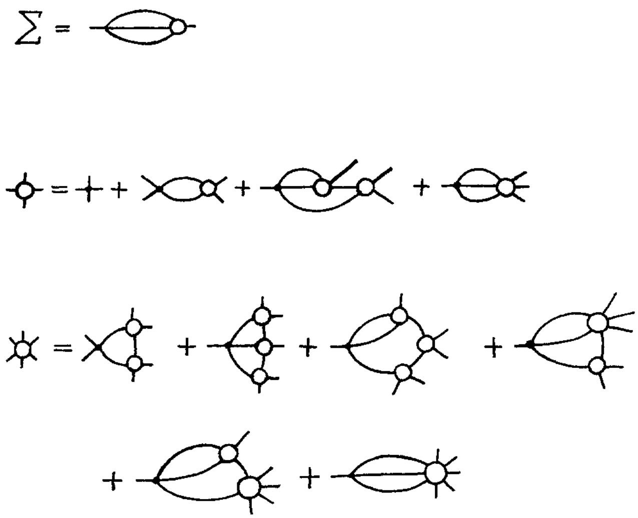


Figure 1. Graphical representation of Green's function equations without G_{2n+1}.

$$\Sigma_i^{\text{reg}}(p) = \lambda \int_{\mu_i^2}^{p^2} d(p'^2) \int_{\mu_i^2}^{p'^2} d(p''^2) \left\{ \sum_{j\neq i} \sum_{k=1}^{N} (G_{2,i} * G_{2,j} * G_{3,ijk} \right.$$

$$* G_{2,k} * G_{3,kji} * G_{2j} + G_{2,j} * G_{2j} * G_{3jjk} * G_{2,k} * G_{3,kii} * G_{2,i})$$

$$\left. + \sum_{j\neq i} G_{2,i} * G_{2,j} * G_{2,j} * G_{4,ijji} \right\} (p'')$$

$$\equiv \lambda([\![[\![G_2 ** 2 * G_3]\!] * G_2 ** 2 * G_3]\!] + [\![G_2 ** 3 * G_4]\!])(p) \quad (1.2b)$$

$$G_{3,ijk} = \lambda^{1/2} [\![G_{2,i} * G_{2,j} * G_{3,ijk}]\!] + (j \leftrightarrow k)$$

$$+ \lambda \sum_{l=i} \sum_{m,n=1}^{N} [\![G_{2,i} * G_{3,ijm} * G_{2,m} * G_{3,lmn} * G_{2,n} * G_{3,nkl} * G_{2,l}]\!] + \cdots \quad (1.2c)$$

Here, $*$ is a shorthand for convolution, and $[\![\ ]\!]$ for regularization. For terms with two-particle thresholds, the integration $\int_{\mu_i^2}^{p^2} d(p'^2)$ in (1.2b) must be taken in the sense of pseudofunctions. We normalize $G_{4,iijj}$ so that $G_{4,iijj}(0,0,0,0) = 1$, therefore our λ corresponds to g^2. [See also Figure 2. For regularization of $G_n (n \geqslant 3)$, see the Appendix.]

If perturbations is assumed, $G_{2n+1} (n = 0,1,2, \ldots)$ are identically zero. We have, however, no a priori reason to assume that the perturbative approach is the only feasible approach. So we consider whether nonzero G_{2n+1}'s are consistent with Green's function equations and the diagonality of self-energy parts (the absence of particle mixing).

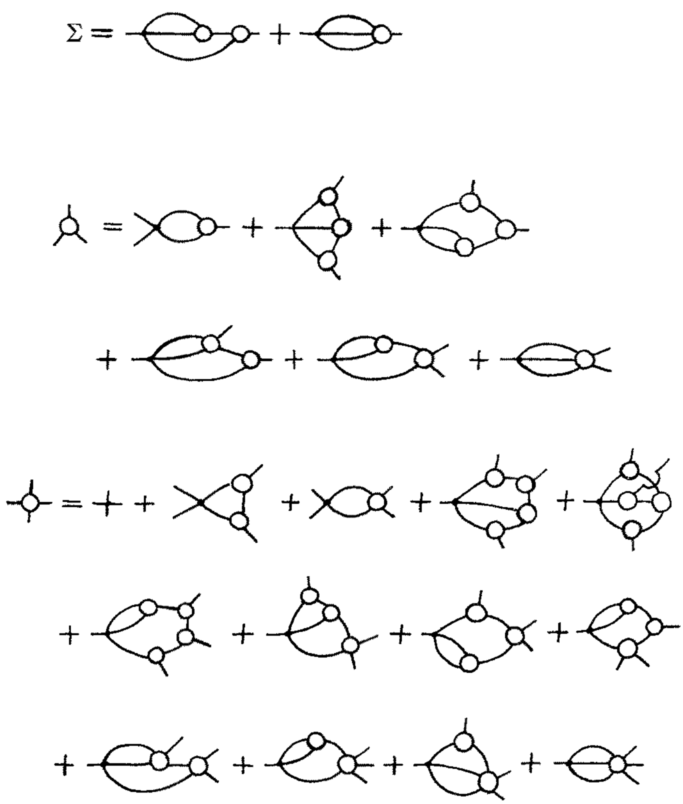


Figure 2. Graphical representation of Green's functions in broken discrete symmetry.

As has been discussed in our previous paper (Yoshimura, 1976), in the nonperturbative approach, one has to give G_4 as an input to the equations for Green's functions. If a G_4 is suitably chosen, one can determine G_2, assuming $G_3 = 0$. What is the situation if, besides G_4, nonzero G_3 or G_5 are given as input?

2. *How many $G_{3,ijk}$'s can be given as nonzero input?*

In this section we consider the combinatorial aspect of the problem. We demand that the self-energy part be diagonal. In other words, we assume the absence of particle mixing. Then only certain $G_{3,ijk}$'s may take nonzero values.

In the case $N = 4$, $G_{3,ijk}$'s with the following combinations of indices can be taken as nonzero input at most (iij, for example, stands for iij, iji, and jii)

$$\{iij, ikl, jjj, jkk, jll\} = I_{4a}$$

or

$$\{iii, ijj, kkk, kll\} = I_{4b}$$

where i, j, k, l are distinct. Otherwise Σ acquires nondiagonal elements.

For $N = 5$, the maximally permissible combinations are

$$\{iij, ikl, jjj, jkk, jll, jmm\} = I_{5a}$$

and

$$\{iii, ijj, kkk, kll, kmm\} = I_{5b}$$

with i, j, k, l, m distinct.

In the case $N = 6$, maximally permissible sets of indices are

$$\{iij, ikl, imn, jjj, jkk, jll, jmm, jnn\} = I_{6a}$$

$$\{iii, ijj, ikk, ill, imm, kkn, klm\} = I_{6b}$$

$$\{iii, ijj, kkk, kll, kmm, knn, lmn\} = I_{6c}$$

$$\{iii, ijj, ikk, klm, lln, mmn, nnn\} = I_{6d}$$

$$\{iii, ijj, ikk, ill, klm, mmn, nnn\} = I_{6e}$$

with i, j, k, l, m, n distinct.

3. *Descending Problem with Given G_3 and G_4*

In this section we consider the problem of finding G_2 when G_3 and G_4 are given. It is convenient to normalize G_4 so that $G_{4,iijj}(0,0,0,0) = 1$. As in our previous paper we assume that G_4 has the following asymptotic behavior:

$$|G_4(p_1, \ldots, p_4)| \leqslant c_1 \frac{|p_1^2|^\alpha \cdots |p_4^2|^\alpha}{(|p_1^2| + \cdots + |p_4^2|)^{4\alpha+\beta}}, \qquad \alpha > 0, \beta \geqslant 0 \tag{3.1}$$

As to G_3, we assume[1]

$$|G_3(p_1, p_2, p_3)| \leqslant c_2 \frac{|p_1^2|^{\alpha'} |p_2^2|^{\alpha'} |p_3^2|^{\alpha'}}{(|p_1^2| + |p_2^2| + |p_3^2|)^{3\alpha'+\beta'}}, \qquad \alpha' > 0, \beta' > 0 \tag{3.2}$$

[1] If α, β, α', β' are not integers, the expression (1.2b) does not have a logarithmic factor and, consequently, is free from ghost. As we do not use perturbative input, arguments based upon Callan–Symanzik-type equations do not apply to the conditions (3.1) and (3.2). The inequalities $\beta > 0$ and $\beta' > 0$ imply convergence of the expressions

$$\frac{d}{d(p^2)} [[G_2 ** 2 * G_3] * G_2 ** 2 * G_3] (p), \frac{d}{d(p^2)} [G_2 ** 3 * G_4] (p)$$

for $|p^2| < \infty$, and

$$0 < \lim_{p^2 \to \infty} G_2(p)(G_2^\circ(p))^{-1} < \infty$$

so that the renormalization constant Z computed from the solution σ^* of equation (3.3) below is finite and the canonical quantization is possible provided the pseudofunction used in the definition of (1.2b) does not dominate with wrong sign in the high-energy limit.

Then the equation to be considered is

$$\Xi[\sigma] \equiv [\Phi - I]\,[\sigma] = 0 \tag{3.3}$$

where

$$\sigma^i = G_{2,i} - G^\circ_{,i} \qquad (G^\circ_{,i}\text{: free propagator with mass } \mu_i)$$

$$\Phi[\sigma] = \lambda([\![G_2 ** 2 * G_3]\!] * G_2 ** 2 * G_3]\!] + [\![G_2 ** 3 * G_4]\!])$$

$$[(G^\circ)^{-1} - \lambda([\![G_2 ** 2 * G_3]\!] * G_2 ** 2 * G_3]\!] + [\![G_2 ** 3 * G_4]\!])]^{-1} \tag{3.4}$$

The map Φ is neither completely continuous, nor contracting, nor monotone. So we appeal to the Newton-Kantorovich scheme. We define the sequence $\{\sigma_n\}$ as follows:

$$\sigma_n = \sigma_{n-1} - [\Xi'(\sigma_{n-1})]^{-1}\Xi[\sigma_{n-1}], \qquad n \in \mathbb{Z}^+ \tag{3.5}$$

Then we can invoke the following theorem (Jankó, 1968).

Theorem 1. Suppose that the following conditions are satisfied:

(1) There exists an element σ_0 such that the Frechet derivative $\Xi[\sigma_0]$ has inverse $(\Xi'[\sigma_0])^{-1}$ that is bounded in norm: $\|\Xi'[\sigma_0]^{-1}\| \leq B_0$. (See footnote 2)

(2) $\|\Xi'[\sigma_0]^{-1}\Xi[\sigma_0]\| \leq \eta_0$

(3) Lipschitz condition
$\|\Phi'[\sigma_1] - \Phi'[\sigma_2]\| \leq K\|\sigma_1 - \sigma_2\| \;\forall \sigma_1, \sigma_2 \in S \equiv \{\sigma \,\|\, \sigma - \sigma_0\| < 2\eta_0\}$

(4) $h_0 \equiv B_0 K \eta_0 < \frac{1}{2}$

Then the equation $\Xi[\sigma] = 0$ has only one solution in S and the series $\{\sigma_n\}$ converges to that solution.

For our equation (3.4) we have the following estimate when $\lambda \ll 1$:

$$B_0 = O(1), \qquad \eta_0 = O(\lambda), \qquad K = O(\lambda)$$

while the integrations contribute finite factors after regularization. Therefore one can expect $h = O(\lambda^2) < \frac{1}{2}$ for sufficiently small λ, i.e., the sequence converges to a locally unique solution

If σ_0, G_3, and G_4 are chosen so as to make η_0 sufficiently small, the theorem holds even if λ is not very small. This is desirable in order that irreducible higher Green's functions do not become too large, because $\|G_{2n+1}\| \sim \lambda^{-n+1}\|G_3\|$.

2 We use the following norm for

$$\|\sigma\| = a_1 \sup_{p^2 \in \mathbb{R}} |\sigma(p)| + a_2 \sup_{p^2 \in \mathbb{R}} |p^2\sigma(p)|$$

This choice corresponds to the canonical quantization.

For existence arguments, we have the following theorem (Fabry, 1973):

Theorem 2. Suppose the following conditions are satisfied:
(1) Map Φ: $S(\sigma_0, \rho) \subset \mathbb{X} \to \mathbb{Y}$ is continuous on $\bar{S}$.
(2) Frechet derivative $\Phi'(\sigma)$ exists for any $\sigma \in S(\sigma_0 \rho)$.
(3) Right inverse $[I - \Phi'(\sigma)]^\dagger$ exists;
(4) There exists a function ψ: $[0, \rho] \to \mathbb{R}^+$ such that
$$[I - \Phi[\sigma]]^\dagger [I - \Phi][\sigma_0] \leq \psi(\|\sigma_0 - \sigma\|) \forall \sigma \in S$$
i.e., $[I - \Phi[\sigma]]^\dagger [I - \Phi][\sigma_0]$ is locally Lipschitzian, and the equation

$$\int_0^r du[\psi(u)]^{-1} = 1$$

has a solution $r^* \in [0, \rho]$.
Then the equation $[I - \Phi][\sigma] = 0$ has a solution $\sigma^* \in S(\sigma_0, \rho)$.

The method of tangent parabola and the method of tangent hyperbola are applicable, too, but we do not repeat the arguments here (see our previous paper, Yoshimura, 1975; also Jankó, 1968).

4. *Ascending Problem*

As has been discussed in our previous paper (Yoshimura, 1976), one cannot determine higher Green's functions without resorting to perturbation theory. Here we consider selection rules of six-point functions in the presence of three-point functions. In the presence of nonzero three-point functions the equations for G_5 and G_6 read

$$G_3 = \lambda^{1/2} [\![G_2 ** 2 * G_3]\!] + (2)\lambda [\![G_2 ** 5 * G_3 ** 3]\!] + (2)\lambda [\![G_2 ** 4 * G_3 * G_4]\!] + \lambda [\![G_2 ** 3 * G_5]\!] \qquad (4.1a)$$

$$\begin{aligned} G_4 = \epsilon &+ \lambda^{1/2} [\![G_2 ** 3 * G_3 ** 2]\!] + \lambda^{1/2} [\![G_2 ** 2 * G_4]\!] \\ &+ (3)\lambda [\![G_2 ** 6 * G_3 ** 4]\!] + (3)\lambda [\![G_2 ** 5 * G_3 ** 2 * G_4]\!] \\ &+ \lambda [\![G_2 ** 4 * G_4 ** 2]\!] + (2)\lambda [\![G_2 ** 4 * G_3 * G_5]\!] \\ &+ \lambda [\![G_2 ** 3 * G_6]\!] \end{aligned} \qquad (4.1b)$$

Here, a number in parentheses stands for the number of topologically distinct graphs (see Figure 2). The "inhomogeneous" term ϵ is 1 when the values of indices are such that bare vertex is present, and is zero otherwise.

In the case $N = 4$, if one takes $G_{4,\,iijj}(i \neq j)$ and G_3's with indices belonging to the set I_{4a} or I_{4b} as nonzero input, the first three terms on the right-hand side of equations (4.1a) do not generate three point functions with other values of indices. On the other hand, even if, for example, input $G_{3,\,iii}$ is zero in the case I_{4a}, three-point functions with indices iii are generated by the first three terms on the right-hand side of equation (4.1a). Therefore the contribution of the last term with $G_{5,\,iiijj}$ must not be zero and must cancel the contributions of the first three terms. This condition can be written in the following form:

$$\lambda^{1/2}[\![G_2 ** 2 * G_3]\!] + (2)\lambda[\![G_2 ** 5 * G_3 ** 3]\!]$$
$$+ (2)\lambda[\![G_2 ** 4 * G_3 * G_4]\!] + \lambda[\![G_2 ** 3 * G_5]\!] = 0 \qquad (4.2)$$

Here indices are suppressed because we shall have to mention equations of the same form but with different values of indices.

The situations are similar in the case $N = 5$.

For the case $N = 6$, the situations are quite different. If one takes $G_{4,\,iijj}$ $(i \neq j)$ and G_3's with indices either I_{6a} or I_{6c} as nonzero input, the first three terms on the right-hand side of equation (4.1a) do not generate three-point functions with other values of indices. But if one takes G_3's with indices I_{6b}, I_{6d}, or I_{6e} as nonzero input, the first three terms on the right-hand side of equation (4.1a) generate three-point functions with other values of indices, so that one gets conditions of the form of equation (4.2) for those values of indices.

The condition (4.2) tells what G_5's must not be zero for a given input, but, as this is an ascending problem, this condition does not determine those G_5's uniquely. Other five-point functions need not be zero but must be orthogonal to $G_2 ** 3$.

Now, let us consider the equation (4.1b). To be specific, take the case I_{6a}. If $G_{4,\,iijj}$ and G_3's with indices I_{6a} are input into equation (4.1b), the second to sixth terms on the right-hand side generate four-point functions with indices such as 1234, 4444, etc., but not such as 1223. Then we get the following conditions:

$$\{\lambda^{1/2}[\![G_2 ** 3 * G_3 ** 2]\!] + \lambda^{1/2}[\![G_2 ** 2 * G_4]\!] + (3)\lambda[\![G_2 ** 6 * G_3 ** 4]\!]$$
$$+ (3)\lambda[\![G_2 ** 5 * G_3 ** 2 * G_4]\!] + \lambda[\![G_2 ** 4 * G_4 ** 2]\!]$$
$$+ (2)\lambda[\![G_2 ** 4 * G_3 * G_5]\!] + \lambda[\![G_2 ** 3 * G_4]\!]\}_{ijkl} = 0 \qquad (4.3)$$

for $ijkl$ such that $G_{4,\,ijkl}$ is assumed to be zero. This equation determines what irreducible six-point functions must not be zero. However, this equation does not determine six-point functions because it is an ascending problem. Alternatively one can assume that $G_{4,\,1234}$ etc. are not zero but input to equation (4.1b), but the minimal set of values of indices for which G_6 must not be zero remains the same.

5. *Descending Problem from Given G_4 and G_5*

In this section, we consider the problem of determining G_2 and G_3 when G_4 and G_5 are given. For this problem, the relevant equations are

$$\lambda G^\circ([\![(G^\circ + \sigma) ** 4 * G_3 ** 2]\!] + [\![(G^\circ + \sigma) ** 3 * G_4]\!])$$
$$\{(G^\circ)^{-1} - \lambda([\![(G^\circ + \sigma) ** 4 * G_3 ** 2]\!] + [\![(G^\circ + \sigma) ** 3 * G_4]\!])\}^{-1} = \sigma \qquad (5.1a)$$

$$\lambda^{1/2}[\![(G^\circ+\sigma) ** 2 * G_3]\!] + (2)\lambda[\![(G^\circ+\sigma) ** 5 * G_3 ** 3]\!]$$
$$+ (2)\lambda[\![(G^\circ+\sigma) ** 4 * G_3 * G_4]\!] + \lambda[\![(G^\circ+\sigma) ** 3 * G_5]\!] = G_3 \quad (5.1b)$$

We write this equation abstractly as follows:

$$\Omega[G_4 G_5; \sigma, G_3] = 0 \quad (5.1')$$

Here, for the Newton-Kantorovich scheme to be applicable, regularization of subintegrations is necessary (see Appendix).
Now, we write the Fréchet derivative Ω as supermatrix whose elements are linear operators:

$$\Omega'[G_4, G_5^\circ, \sigma^\blacktriangle, G_3^\blacktriangle; .,.] = \begin{bmatrix} \Omega'_{11} & \Omega'_{12} \\ \Omega'_{21} & \Omega'_{22} \end{bmatrix} \quad (5.2a)$$

$$\Omega'_{11} = \lambda[(4)[\![(G^\circ+\sigma^\blacktriangle) ** 3 * \cdot * G_3^\blacktriangle ** 2]\!] + (3)[\![(G^\circ+\sigma^\blacktriangle) ** 2 * \cdot * G_4]\!]]$$
$$\{(G^\circ)^{-1} - \lambda[[\![(G^\circ+\sigma^\blacktriangle) ** 4 * G_3^\blacktriangle ** 2]\!] + [\![(G^\circ+\sigma^\blacktriangle) ** 3 * G_4]\!]]\}^{-2} - I$$
$$(5.2b)$$

$$\Omega'_{12} = \lambda(2)[\![(G^\circ+\sigma^\blacktriangle) ** 4 * G_3^\blacktriangle * \cdot\ ]\!]$$
$$\{(G^\circ)^{-1} - \lambda[\![(G^\circ+\sigma^\blacktriangle) ** 4 * G_3^\blacktriangle ** 2]\!] + [\![(G^\circ+\sigma^\blacktriangle) ** 3 * G_4]\!]\}^{-2}$$
$$(5.2c)$$

$$\Omega'_{21} = (2)\lambda^{1/2}[\![(G^\circ+\sigma^\blacktriangle) * \cdot * G_3]\!] + (10)\lambda[\![(G^\circ+\sigma^\blacktriangle) ** 4 * \cdot * G_3^\blacktriangle ** 3]\!]$$
$$+ (8)\lambda[\![(G^\circ+\sigma^\blacktriangle) ** 3 * \cdot * G_3^\blacktriangle * G_4]\!] + (3)\lambda[\![(G^\circ+\sigma^\blacktriangle) ** 2 * \cdot * G_5]\!]$$
$$(5.2d)$$

$$\Omega'_{22} = \lambda^{1/2}[\![(G^\circ+\sigma^\blacktriangle) ** 2 * \cdot\ ]\!] + (9)\lambda[\![(G^\circ+\sigma^\blacktriangle) ** 5 * G_3 ** 2 * \cdot\ ]\!]$$
$$+ (2)\lambda[\![(G^\circ+\sigma^\blacktriangle) ** 4 * \cdot * G_4]\!] - I \quad (5.2e)$$

For sufficiently small λ and $[\![G^\circ ** 3 * G_5]\!]$ with sufficiently small norm, one gets the following estimate:

$$B = O(1), \qquad \eta_0 = O(\lambda^{1/2}), \qquad K = O(\lambda^{1/2}), \qquad h_0 = B\eta_0 K = O(\lambda) \quad (5.3)$$

i.e., the solution is locally unique, if one begins with the zeroth approximation $\sigma_0 = 0$, $G_{3,ijk,0} = 0$. Because of the "inhomogenuity" of equation (4.3), there is no trivial solution.

One cannot descend from a given set of G_5 and G_6 by the Newton-Kantorovich scheme because of the presence of the inhomogeneous term ϵ on the right-hand side of equation (4.1b). However, if a set of $G_n (n = 2,3,\ldots,6)$ are given, one cannot change $G_m (m = 2,3,4)$ continuously without affecting G_5 and G_6. In other words, if two or more sets $\{G_5, G_6\}$, corresponding to the same set $G_m (m = 2,3,4)$ exist at all, norms of their differences must be finite, because of the local uniqueness theorem.

If we do not demand that $G_{4,1234}$, etc., be zero, we get "homogeneous" equations for these G_4's:

$$G_{4,1234} = \{\lambda^{1/2}[\![(G^\circ + \sigma^\blacktriangle) ** 3 * G_3 ** 2]\!] + \cdots + \lambda[\![(G^\circ + \sigma^\blacktriangle) ** 3 * G_6]\!]\}_{1234} \qquad (5.4)$$

etc., as these equations do not involve those G_6's that appear in the "inhomogeneous" equations for $G_{4,iijj}$, one can solve (at least in principle) descending problems from $G_{4,iijj}$, G_5, and G_6 that appear in equation (5.4).

If a descending problem involves equations of the form of equation (4.2) and/or (4.3), the Newton–Kantorovich scheme is not applicable because the Fréchet derivative is not invertible. If G_5 appearing in equation (5.1b) are solutions of an ascending problem involving equation (4.2), then the solution of equation (5.1) such that G_3's with appropriate values of indices are zero exist.

6. *Interaction* $\sum_{i,j,k,l \text{ pairwise distinct}} \phi_i\phi_j\phi_k\phi_l$

In this section, we consider the following interaction Lagrangian:

$$\mathscr{L}_I' = g' \sum_{i,j,k,l \text{ pairwise distinct}} \phi_i\phi_j\phi_k\phi_l$$

In the case $N = 4$, if $G_{3,123}$ is the only zonzero input three-point functions, the self-energy parts are not affected at all, but the first three terms on the right-hand side generate three-point functions with indices 114, 224, and 334, which must be cancelled by the last term involving irreducible five-point functions. If other input G_3's are not zero, G_3's contribute to the self-energy parts. Even if nonzero G_3's are so chosen that the self-energy is diagonal (i.e., there is no particle mixing) the first three terms on the right-hand side of equation (4.1a) generate many other three-point functions, which must be cancelled by the contribution of irreducible five-point functions. This requirement determines a minimal set of indices for which irreducible five-point functions must not be zero when a set of nonzero three-point functions are given as input.

The situation is quite different in the case $N = 5$. In this case, either the contribution of G_3's to the self-energy parts is identically zero or particle mixing occurs, depending on the set of indices for which G_3's are not zero. In other words, there is no set of G_3's that contributes to the self-energy parts without causing particle mixing. When the input set of G_3's is such that G_3's do not contribute to the self-energy parts, the first three terms on the right-hand side of equation (4.1a) generate three-point functions with only such values of indices for which input G_3's are zero. The second to sixth terms on the right-hand side of equation (4.1b) do not generate four-point functions with indices such as 1223 etc., but 1234 etc. As $G_{4,1234}$ etc. do not cause particle mixing, we need not assume them to be zero.

The situation is quite similar for the case $N = 6$.

7. *Concluding Remarks*

If a Lagrangian (or Hamiltonian) is invariant only under a discrete group, Goldstone's theorem does not apply, so that solutions of the equations for many-point functions need not inherit the invariance of the Lagrangian. As transition between a "normal" symmetry-preserving solution and a symmetry-breaking solution is not defined in quantum field theory, the question of stability of the latter does not arise. This is an essential difference between the quantum field theory and the many-body problem.

Contrary to the so-called bootstrap mechanism, our equations do not make sense if $\lambda = 0$. Interaction must be triggered by $\lambda \neq 0$. If one demands that $\|G_{2n+1}^{\text{irred}}\| \to 0$ as $n \to \infty$, symmetry-breaking solutions are excluded. As our input involves many parameters with dimensions, arguments based on scale invariance do not apply, while the renormalization group equations are trivial unless extra input is assumed.

As far as lower Green's functions are concerned, G_3 and G_4, or G_4 and G_5 or G_4, G_5, and some G_6 can be given as arbitrary input subject to certain restrictions, though G_{2n+1} must be identically zero in perturbation theory.

Though the integrals

$$[G_2 ** 4 * G_3 ** 2](p), \qquad [G_2 ** 3 * G_4](p), \qquad (|p^2| < \infty)$$

converge for G_3 and G_4 satisfying the conditions (3.1) with $\alpha + \beta > 1$ and (3.2) and consequently $\delta\mu$ and Z are finite, one cannot use the conventional renormalization procedure for the descending problem (Yoshimura, 1975a). Therefore, it is not possible to compute dynamically generated masses even if vertex parts are such that the integrals in equation (1.2b) converge. Given a Langrangian, the equations for many-point functions are derived involving masses as arbitrary parameters, even if G_3's and G_4's are given. (Perturbation theory does not enable one to compute dynamically generated masses, either).

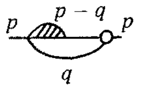
In the presence of particle mixing, one cannot separate a pole-free object σ from two-point functions, so that one does not have a set of suitable objects to be found as a fixed point of operator equations in direct product of Banach spaces. Therefore, if a given set of G_3's is such that particle mixing occurs, one cannot descend from G_3's and G_4's by any known method, though solutions may or may not exist.

If a continuous symmetry group is already (spontaneously or nonspontaneously) broken into a product of a continuous group and a discrete group, the solution of Green's function equations need not inherit the invariance of the Lagrangian under the discrete group, without generating Goldstone particles.

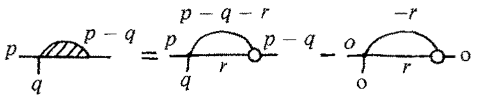
Appendix: Regularization

Lest the right-hand side of equation (3.4) should have poles even when σ is not a solution of equation (3.3), one has to regularize the self-energy parts by the following substitution (Taylor, 1968):

$$\Sigma_i(p) \to \Sigma_i^{\text{reg}}(p) \equiv \int_{\mu_i^2}^{p^2} d(p'^2) \int_{\mu_i^2}^{p'^2} d(p''^2) \frac{d^2 \Sigma_i^r(p'')}{d(p''^2)^2} \tag{A1a}$$

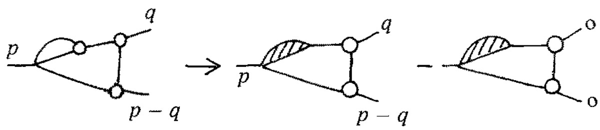
$\Sigma_i^r(p) \equiv$ 

$+ [G_2 ** 3 * G_4](p)$ (A1b)



(A1c)

not by renormalization constants $\delta\mu$ and Z. For diagrams with three or more external lines, one must use substitutions, e.g.,



(A2)

With  as above.

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